1. Carefully define the terms marked in bold type. [2 each]
   a) A measure $\mu$ on a measurable space $(X, \mathcal{F})$.
   b) A measurable function $f : X \rightarrow Y$ of measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$.
   c) A $\sigma$-finite measure space $(X, \mathcal{F}, \nu)$.
   d) A Banach space $M$.
   e) A (Hilbert space) basis of a Hilbert space $H$.
   f) The Fourier transform $\mathcal{F}(f)$ of $f \in L^1(\mathbb{R}, \mu)$.
   g) The convolution $f * h$ of $f, h \in L^1(\mathbb{R}, \mu)$.

2. Carefully state, with details, the following results. [2 each]
   a) The Cauchy-Schwarz Inequality.
   b) Stone-Weierstrass Theorem for $C(X, \mathbb{C})$, where $X$ is a compact Hausdorff (or metric) space.
   c) The Hölder Inequality.
   d) The Monotone Convergence Theorem.

3. [3]
   a) Define a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous ae.
   b) Show, or give a counterexample to, the following (provide details of your reasoning): If $f : [0, 1] \rightarrow \mathbb{R}$ measurable with $f = 0$ ae, then $f$ is continuous ae.

4. Let $(M, \rho)$ be a metric space. Assume $S$ a compact subset of $M$, $p \in M$ and $d = \inf_{a \in S} \rho(a, p)$. Show there is an $m \in S$ with $\rho(m, p) = d$. [3]

5. Let $(X, \mathcal{F}, \nu)$ be a measure space and $f : X \rightarrow \mathbb{R}$ a measurable function, $f \geq 0$ ae. Prove if $\int_X f d\nu = 0$ then $f = 0$ ae. [3]

6. [2]
   a) Show $\|f\| = \int_{[0,1]} |f| \, d\mu$ defines a norm on $C([0,1], \mathbb{R})$.
   b) Show this normed space is not complete (as a metric space). [2]
7. 
   a) Prove, quoting results you use, that \( \sum_{n=0}^{\infty} (2/3)^n \exp(2\pi int) \) defines a continuous function \( f \) of \( t \) on \([0, 1]\). [3] 
   b) Show \( \mathcal{C}([0, 1]) \) is contained in \( L^2([0, 1], \mu) \). [1] 
   c) For this \( f \) compute \( \|f\|_2 \). [2] 

Further questions for those graduate students writing the preliminary exam.

8. Let \( \{e_n | n \in I\} \) be a (Hilbert space) basis for a Hilbert space \( \mathcal{H} \). Show there is a Hilbert space isomorphism (a surjective linear isometry) \( \mathcal{H} \rightarrow l^2(I) \). [4] 

9. Show \( \{\exp(2\pi int)\}_{n \in \mathbb{Z}} \) is a Hilbert space basis of \( L^2([0, 1], \mu) \). You may assume that \( \mathcal{C}([0, 1]) \) is dense in \( L^2([0, 1], \mu) \) (in the \( \|\|_2 \) norm). [3] 

10. Let \( f \in L^1(\mathbb{R}, \mu), \epsilon > 0 \), and \( A_n = \{x \in \mathbb{R} | |x| \geq n\} \) for \( n \in \mathbb{N} \). Show there is an \( N_0 \in \mathbb{N} \) such that \( \int_{A_n} |f| < \epsilon \) for all \( n > N_0 \). [3]