1. Random Number Generation

In the following assume you have a routine to generate an infinite series of standard uniform variates $U_1, U_2, \ldots$.

a) Devise an inversion method for simulating $n$ realizations from the Cauchy distribution with probability density function,

$$f_X(x) = \frac{1}{\pi(1 + x^2)} \quad -\infty < x < \infty.$$ 

Write down your algorithm. [5]

(Hint: $\int \frac{1}{1+x^2} = \tan^{-1}(x) + C$)

b) Devise a rejection method for simulating $n$ realizations from the “half-normal” density, given by

$$f_X(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} \text{ for } x \geq 0,$$

using $g(x) = Me^{-x}$ as the enveloping function. Write down your algorithm. Find $M^*$, the most efficient value of $M$. What is the theoretical probability of rejection? [5]
2. Antithetic and Control Variates

Consider using simulation to estimate the integral

\[ \theta = \mathbb{E}[e^U] = \int_0^1 e^x \, dx \]

where \( U \sim U[0, 1] \).

a) Explain the principles of using antithetic variates for reducing the variance of an estimate. [3]

b) Show that \( \text{Var}[\theta_a] = 0.1210 \) where

\[ \theta_a = \frac{e^{U_1} + e^{U_2}}{2} \]

and \( U_1, U_2 \sim U[0, 1] \) [3]

c) Show that the variance of the antithetic variate \( \theta_b \) is lower where

\[ \theta_b = \frac{e^U + e^{1-U}}{2} \] [3]

d) Consider the estimation of \( \theta \) by \( \theta_c = e^U \) where \( U \sim U[0, 1] \). Show how you can use the control variate \( C = U \) with \( \theta_c \) to produce an estimate of \( \theta \). Show that the variance of your new estimator is lower than that of \( \theta_c \). [5]

3. MCMC Inference using Gibbs

Consider the following posterior distribution:

\[ \pi(\theta, \lambda, n|\alpha, \beta, \gamma) \propto \binom{n}{\theta} \lambda^{\theta+\alpha-1} (1-\lambda)^{n-\theta+\beta-1} \gamma^n \frac{\gamma^\theta}{n!} e^{-\gamma}. \]

(Note that here \( \alpha, \beta, \gamma \) are essentially the data).

a) Give the conditional posterior distribution \( \pi(\theta|\lambda, n, \alpha, \beta, \gamma) \). [2]

b) Give the conditional posterior distribution \( \pi(\lambda|\theta, n, \alpha, \beta, \gamma) \). [3]

c) Give the conditional posterior distribution \( \pi(n|\lambda, \theta, \alpha, \beta, \gamma) \). [5]

(HINT: all the above can be found with standard forms.)

d) Describe a Gibbs sampler for sampling from the posterior distribution given above. [2]
4. MCMC Inference using Metropolis-Hastings

Describe the following within the context of using MCMC to explore some multivariate target distribution:

a) The Metropolis Hastings algorithm, along with examples of two types of proposal update. [5]

b) The advantages and disadvantages of using the Gibbs sampler as opposed to a random walk Metropolis Hastings update for a single parameter [3]

c) The reasons and circumstances why we might, and might not, wish to use a block Metropolis Hastings update for a set of parameters rather than a series of single updates. [3]

d) The advantages and disadvantages of running a number of short-running Metropolis Hastings MCMC chains as opposed to a long single chain. [3]